Fusion in Virasoro logarithmic models and the Kazhdan–Lusztig correspondence

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Introduction.

LCFTs most naturally appear as

- a scaling limit of nonlocal lattice models (Pearce, Rasmussen, and Zuber, 2006);
- and in quantum chains with a nondiagonalizable Hamiltonian (Read, Saleur, 2007).

Generally speaking, a CFT appearing at the limit depends

- on a way of taking the limit and
- on chosen boundary conditions.

Introduction.

Generally speaking, a CFT appearing at the limit depends on a way of taking the limit and on chosen boundary conditions.

For proper choice of boundary conditions

the lattice models of PRZ —-> Log models $\mathcal{WLM}(p,p')$ with

the *triplet* $\mathcal{W}_{p,p'}$-algebra of symmetry

(FGST, 2006; and for $p'=1$ by Kausch and FHST)

The chiral algebra $\mathcal{W}_{p,p'}$ is an extension of the vacuum module of the Virasoro algebra $\mathcal{V}_{p,p'}$ with the central charge $c_{p,p'} = 13 - 6p/p' - 6p'/p$ by a triplet of the Virasoro primary fields with conformal dimension $\Delta_{1,3}$.

Introduction.

The most investigated models are $\mathcal{WLM}(p,1)$.

For this set of models, the representation categories of

- the triplet algebra $\mathcal{W}_p$
- and of the finite-dimensional quantum group $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$ with $\mathfrak{q} = e^{i\pi/p}$ (FGST, 2005)

are **equivalent** as *tensor categories* (FGST2 (2005), for $p = 2$; and by Nagatomo and Tsuchiya (2009), Adamovic and Milas (2009), as abelian categories)

Introduction.

Categories of $\mathcal{W}_p$- and $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$-modules are **equivalent** as *tensor categories*

This is the manifestation of the Kazhdan–Lusztig duality:

(1) there is a one-to-one correspondence between representations;
(2) fusion rules of a conformal model can be calculated by tensor products of a quantum group representations
(3) and the modular group action generated from chiral characters coincides with the one on the center of the corresponding quantum group.

In the logarithmic models $\mathcal{WLM}(1,p)$, the Kazhdan–Lusztig duality is presented in its **full strength**.

Introduction.

For general coprime p and p', the models $\mathcal{WLM}(p,p')$ also demonstrate the Kazhdan–Lusztig duality with the quantum group

$$\mathfrak{g}_{p,p'} = \frac{\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2) \otimes \overline{\mathfrak{U}}_{\mathfrak{q}'} s\ell(2)}{\textit{Hopf ideal}}, \quad \mathfrak{q} = \mathrm{e}^{i\pi/p} \;\; \textit{and} \;\; \mathfrak{q}' = \mathrm{e}^{i\pi/p'}$$

but relation between the $\mathfrak{g}_{p,p'}$ and the $\mathcal{W}_{p,p'}$ algebra is more subtle.

There is *no* one-to-one correspondence between representations but

- the modular group action on the center of $\mathfrak{g}_{p,p'}$ (FGST, 06) **coincides** with
- the one on chiral characters in the $\mathcal{W}_{p,p'}$ theory

and an open question about fusion...

Introduction.

Other choice of boundary conditions in the lattice models of PRZ

—-> Log models $\mathcal{LM}(p,p')$ with the Virasoro symmetry $\mathcal{V}_{p,p'}$.

Fusion rules for these models were calculated in

(1) Pearce, Rasmussen, 2007 (lattice approach)

and for some cases in

(2) Gaberdiel, Kausch, 1996;
Eberle, Flohr, 2006 (Gaberdiel–Kausch–Nahm algorithm)

(3) Read, Saleur, 2007 — using quantum-group symmetries in XXZ models at a root of unity and fusion procedure of Temperley–Lieb algebra representations.

Introduction.

- We propose using the Kazhdan–Lusztig duality in calculating the fusion rules for the subset $\mathcal{LM}(1,p)$ of the $\mathcal{LM}(p,p')$ models.

Introduction.

We construct a quantum group dual to the Virasoro algebra $\mathcal{V}_p$ from $\mathcal{LM}(1,p)$ as an extension of $\overline{\mathfrak{U}}_{\mathfrak{q}}s\ell(2)$ dual to the triplet algebra $\mathcal{W}_p$.

- This quantum group is the Lusztig limit $\mathcal{L}\mathfrak{U}_{\mathfrak{q}}s\ell(2)$ of the usual quantum $s\ell(2)$ as $\mathfrak{q} \to e^{i\pi/p}$ and
- has the set of irreducible representations $\mathfrak{X}^{\alpha}_{s,r}$, where $1 \leqslant s \leqslant p$ and $\alpha = \pm$ are $\overline{\mathfrak{U}}_{\mathfrak{q}}s\ell(2)$ h.w. parameters and $\frac{r-1}{2}$, $r \in \mathbb{N}$, is the $s\ell(2)$ spin.
- The module $\mathfrak{X}^{\alpha}_{s,r}$ is a tensor product of s-dimensional irreducible $\overline{\mathfrak{U}}_{\mathfrak{q}}s\ell(2)$- and r-dimensional irreducible $s\ell(2)$-modules.
- To each $\mathfrak{X}^{\alpha}_{s,r}$, a projective cover $\mathfrak{P}^{\alpha}_{s,r}$ corresponds and $\mathfrak{P}^{\alpha}_{p,r} = \mathfrak{X}^{\alpha}_{p,r}$.

Introduction.

We construct a quantum group dual to the Virasoro algebra $\mathcal{V}_p$ from $\mathcal{LM}(1,p)$ as an extension of $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ dual to the triplet algebra $\mathcal{W}_p$.

- The set of irreducible $\mathcal{X}^{\alpha}_{s,r}$ and projective modules $\mathcal{P}^{\alpha}_{s,r}$ is closed under tensor products.
- the Pearce–Rasmussen fusion of irreducible and logarithmic $\mathcal{V}_p$-representations **coincides** with tensor products of $\mathcal{L}\mathcal{U}_{\mathfrak{q}} s\ell(2)$ irreducible and projective modules.

Theorem. *The tensor products between irreducible $\mathcal{L}\mathcal{U}_{\mathfrak{q}} s\ell(2)$-modules are*

$$\mathcal{X}^{\alpha}_{s_1,r_1} \otimes \mathcal{X}^{\beta}_{s_2,r_2} = \bigoplus_{\substack{r=|r_1-r_2|+1\\ step=2}}^{r_1+r_2-1} \Bigg(\bigoplus_{\substack{s=|s_1-s_2|+1\\ step=2}}^{\substack{\min(s_1+s_2-1,\\ 2p-s_1-s_2-1)}} \mathcal{X}^{\alpha\beta}_{s,r} + \bigoplus_{\substack{s=2p-s_1-s_2+1\\ step=2}}^{p-\gamma_2} \mathcal{P}^{\alpha\beta}_{s,r} \Bigg)$$

between the irreducible and projective modules are

$$\mathcal{X}^{\alpha}_{s_1,r_1} \otimes \mathcal{P}^{\beta}_{s_2,r_2} = \bigoplus_{\substack{r=|r_1-r_2|+1\\ step=2}}^{r_1+r_2-1} \Bigg(\bigoplus_{\substack{s=|s_1-s_2|+1\\ step=2}}^{\substack{\min(s_1+s_2-1,\\ 2p-s_1-s_2-1)}} \mathcal{P}^{\alpha\beta}_{s,r} + 2 \bigoplus_{\substack{s=2p-s_1-s_2+1\\ step=2}}^{p-\gamma_2} \mathcal{P}^{\alpha\beta}_{s,r} \Bigg) + 2 {\bigoplus_{\substack{r=|r_1-r_2|\\ step=2}}^{r_1+r_2}}' \bigoplus_{\substack{s=p-s_1+s_2+1\\ step=2}}^{p-\gamma_1} \mathcal{P}^{-\alpha\beta}_{s,r},$$

and between the projective modules are

$$\mathcal{P}^{\alpha}_{s_1,r_1} \otimes \mathcal{P}^{\beta}_{s_2,r_2} = 2 \bigoplus_{\substack{r=|r_1-r_2|+1\\ step=2}}^{r_1+r_2-1} \Bigg(\bigoplus_{\substack{s=|s_1-s_2|+1\\ step=2}}^{\substack{\min(s_1+s_2-1,\\ 2p-s_1-s_2-1)}} \mathcal{P}^{\alpha\beta}_{s,r} + 2 \bigoplus_{\substack{s=2p-s_1-s_2+1\\ step=2}}^{p-\gamma_2} \mathcal{P}^{\alpha\beta}_{s,r} \Bigg)$$

$$+2 {\bigoplus_{\substack{r=|r_1-r_2|\\ step=2}}^{r_1+r_2}}' \Bigg(\bigoplus_{\substack{s=|p-s_1-s_2|+1\\ step=2}}^{\substack{\min(p-s_1+s_2-1,\\ p+s_1-s_2-1)}} \mathcal{P}^{-\alpha\beta}_{s,r} + 2 \bigoplus_{\substack{s=\min(p-s_1+s_2+1,\\ p+s_1-s_2+1)}}^{p-\gamma_1} \mathcal{P}^{-\alpha\beta}_{s,r} \Bigg) + 4 {\bigoplus_{\substack{r=|r_1-r_2|-1\\ step=2}}^{r_1+r_2+1}}'' \bigoplus_{\substack{s=s_1+s_2+1\\ step=2}}^{p-\gamma_2} \mathcal{P}^{\alpha\beta}_{s,r},$$

where we set $\gamma_1 = (s_1 + s_2 + 1) \bmod 2,\ \gamma_2 = (s_1 + s_2 + p + 1) \bmod 2$.

Introduction. We thus have

- the $\mathcal{L}\mathfrak{U}_{\mathfrak{q}}s\ell(2)$ representation category is **equivalent** as a tensor category to the category of Virasoro algebra representations appearing in $\mathcal{LM}(1,p)$.

Irreducible and projective modules are identified in the following way

$$\begin{aligned}
&\mathcal{X}^{+}_{s,2r-1} \to (2r-1,s), && \mathcal{X}^{-}_{s,2r} \to (2r,s),\\
&\mathcal{P}^{+}_{s,2r-1} \to \mathcal{R}^{p-s}_{2r-1}, && \mathcal{P}^{-}_{p-s,2r} \to \mathcal{R}^{s}_{2r}, \quad 1 \leqslant s \leqslant p, \quad r \geqslant 1,
\end{aligned}$$

where (r,s) are the irreducible Virasoro modules with the heighest weights

$$\Delta_{r,s} = ((pr-s)^2 - (p-1)^2)/4p$$

and the $\mathcal{R}^s_r$ are logarithmic Virasoro modules from $\mathcal{LM}(1,p)$.

Quantum groups as centralizers of chiral algebras.

The QGs dual to Log models $\mathcal{LM}(1,p)$ as well as $\mathcal{WLM}(1,p)$ can be constructed in the Coulomb gas picture

$$\varphi(z)\varphi(w) = \log(z-w)$$

with the energy-momentum tensor

$$T = \tfrac{1}{2}\partial\varphi\partial\varphi + \tfrac{\alpha_0}{2}\partial^2\varphi,$$

where the background charge $\alpha_0 = \alpha_+ + \alpha_- = \sqrt{2p} - \sqrt{2/p}$.

- Chiral algebras and corr. QGs are mutual **maximal centralizers** of each other on a chiral space of states.
- There are two screening operators (“long” and “short”)

$$e = \oint \mathrm{e}^{\sqrt{2p}\,\varphi(z)}dz \quad \textit{and} \quad F = \oint \mathrm{e}^{-\sqrt{\frac{2}{p}}\,\varphi(z)}dz$$

  commuting with $\mathcal{V}_p$.

The centralizer of $\mathcal{W}_p$.

The quantum group $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$ **commutes** with the triplet algebra $\mathcal{W}_p$ action on "full" chiral space of states.

- the chiral algebras $\mathcal{W}_p$ realized in the $\mathcal{WLM}(1,p)$ models admit $s\ell(2)$-action by symmetries:

$$W^-(z) := \mathrm{e}^{-\sqrt{2p}\varphi}(z), \quad W^0(z) := [e, W^-(z)], \quad W^+(z) := [e, W^0(z)],$$

  where e is the long screening operator $\oint \mathrm{e}^{\sqrt{2p}\varphi} dz$ (see FHST, 2004)
- the short screening F commutes with the chiral algebra $\mathcal{W}_p$
- and generates the lower-triangular part of the $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$ with the relation $F^p = 0$.

The centralizer of $\mathcal{W}_p$.

Construction of $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$:

(1) Hopf algebra of the short screening $F = \oint \mathrm{e}^{-\sqrt{\frac{2}{p}}\varphi(z)} dz$ and the Kartan $K = \mathrm{e}^{-i\pi\alpha_-\varphi_0}$, where φ_0 is the zero-mode of $\partial\varphi(z)$.

Hopf-algebra structure is found from the action of these operators on fields: *comultiplication* is calculated from the action of F and K on OPE of fields.

(2) Drinfeld double —-> contour-removal operator E (dual to F) and additional Kartan $\bar{K}$.

(3) the quantum group $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$ is realized as a quotient of the Drinfeld double

The centralizer of $\mathcal{W}_p$.

The "restricted" quantum group $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ with $\mathfrak{q} = e^{i\pi/p}$ and the three generators E, F, and K satisfying the standard relations

$$KEK^{-1} = \mathfrak{q}^2 E, \quad KFK^{-1} = \mathfrak{q}^{-2} F, \quad [E, F] = \tfrac{K - K^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}},$$

with some additional constraints,

$$E^p = F^p = 0, \quad K^{2p} = \mathbf{1},$$

and the Hopf-algebra structure is given by

$$\Delta(E) = \mathbf{1} \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes \mathbf{1}, \quad \Delta(K) = K \otimes K,$$

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1},$$

$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = 1.$$

The centralizer of $\mathcal{V}_p$.

To construct a QG dual to the Virasoro algebra $\mathcal{V}_p$ from $\mathcal{LM}(1,p)$, we first note that

- Irreducible representations of the triplet algebra $\mathcal{W}_p$ admit two commuting actions, $s\ell(2)$- and $\mathcal{V}_p$-actions (FGST, 2006):

  Considering the deformation

$$F_\epsilon = \oint \mathrm{e}^{(-\sqrt{\frac{2}{p}}+\epsilon)\varphi(z)}dz \qquad \longrightarrow \qquad f = \lim_{\epsilon\to 0}\frac{F_\epsilon^p}{\epsilon},$$

  The operators $e = \oint \mathrm{e}^{\sqrt{2p}\varphi}dz$ and f generate the usual $s\ell(2)$.

The centralizer of $\mathcal{V}_p$.

We thus have the $s\ell(2)$-generators:

$$h = \frac{1}{\sqrt{2p}}\,\varphi_0, \qquad e = \oint \mathrm{e}^{\sqrt{2p}\,\varphi(z)}dz, \qquad \textit{and} \qquad f = \lim_{\epsilon \to 0} \frac{F^p_\epsilon}{\epsilon}.$$

- Invariants of the $s\ell(2)$-action is the universal enveloping of the Virasoro algebra $\mathcal{V}_p$.

These points suggest a construction of the maximal centralizer for $\mathcal{V}_p$ as an extension of the centralizer $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$ for the triplet algebra $\mathcal{W}_p$ by the $s\ell(2)$ triplet (e, h, f).

The centralizer of $\mathcal{V}_p$.

To obtain a Hopf-algebra structure on $\mathcal{L}\mathcal{U}_{\mathfrak{q}} s\ell(2)$, we use the purely algebraic approach following Lusztig:

- the quantum group $\mathcal{L}\mathcal{U}_{\mathfrak{q}} s\ell(2)$ is a limit of the quantum group $U_{\mathfrak{q}}(s\ell(2))$ as $\mathfrak{q} \to \mathrm{e}^{\frac{\imath\pi}{p}}$.
- There is an evident limit in which E^p, F^p and K^p become central

The centralizer of $\mathcal{V}_p$.

To obtain a Hopf-algebra structure on $\mathcal{L}\mathcal{U}_{\mathfrak{q}} s\ell(2)$, we use the purely algebraic approach following Lusztig:

- but we consider another limit in which the relations

  $$E^p = F^p = 0, \quad K^{2p} = 1$$

  are imposed but the generators

  $$e = \frac{E^p}{[p]!} \qquad \textit{and} \qquad f = \frac{F^p}{[p]!}, \qquad [n] = \frac{\mathfrak{q}^n - \mathfrak{q}^{-n}}{\mathfrak{q} - \mathfrak{q}^{-1}},$$

  are kept in the limit.

In the limit $\mathfrak{q} \to e^{\frac{i\pi}{p}}$, we have $[p]! = 0$ and the ambiguity $\frac{0}{0}$ is solved in such a way that the e and f become generators of the ordinary $s\ell(2)$.

The centralizer of $\mathcal{V}_p$.

We thus obtain a Hopf algebra $\mathcal{L}\mathfrak{U}_{\mathfrak{q}} s\ell(2)$ that contains the quantum group $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$ as a Hopf ideal and the quotient is the $U(s\ell(2))$, the universal enveloping of the $s\ell(2)$.

The centralizer of $\mathcal{V}_p$.

The Hopf-algebra structure on $\mathcal{L}\mathfrak{U}_{\mathfrak{q}}s\ell(2)$ is the following. The defining relations between the E, F, and K generators are the same as in $\overline{\mathfrak{U}}_{\mathfrak{q}}s\ell(2)$ and the usual $s\ell(2)$ relations between the e, f, and h:

$$[h,e]=e,\qquad [h,f]=-f,\qquad [e,f]=2h,$$

and the "mixed" relations

$$[h,K]=0,\qquad [E,e]=0,\qquad [K,e]=0,\qquad [F,f]=0,\qquad [K,f]=0,$$
$$[F,e]\sim(\mathfrak{q}K-\mathfrak{q}^{-1}K^{-1})\,E^{p-1},$$
$$[E,f]\sim(\mathfrak{q}K-\mathfrak{q}^{-1}K^{-1})\,F^{p-1},$$
$$[h,E]=\frac{1}{2}EA,\quad [h,F]=-\frac{1}{2}AF,$$

where A is a projector.

The centralizer of $\mathcal{V}_p$.

The comultiplication in $\mathcal{L}\mathcal{U}_{\mathfrak{q}} s\ell(2)$ is

$$\Delta(e) = e \otimes 1 + K^p \otimes e + \tfrac{1}{[p-1]!} \sum_{r=1}^{p-1} \frac{\mathfrak{q}^{r(p-r)}}{[r]} K^p E^{p-r} \otimes E^r K^{-r},$$

$$\Delta(f) = f \otimes 1 + K^p \otimes f + \tfrac{(-1)^p}{[p-1]!} \sum_{s=1}^{p-1} \frac{\mathfrak{q}^{-s(p-s)}}{[s]} K^{p+s} F^s \otimes F^{p-s},$$

an explicit form of $\Delta(h) = \frac{1}{2}[\Delta(e), \Delta(f)]$ is very bulky and we do not give it here.

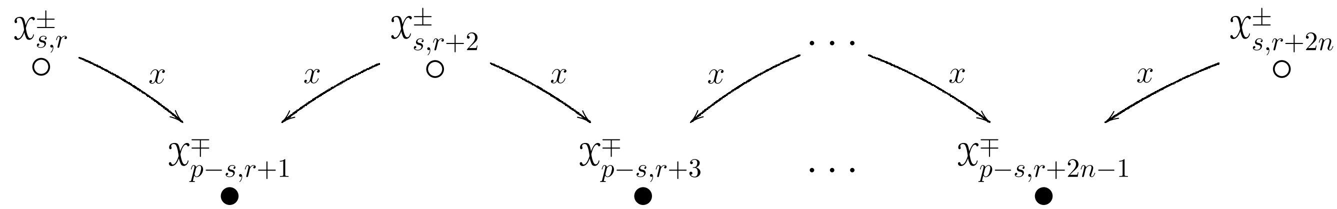
The antipode S and the counity ϵ are

$$S(e) = -K^p e, \qquad S(f) = -K^p f, \qquad S(h) = -h,$$

$$\epsilon(e) = \epsilon(f) = \epsilon(h) = 0.$$

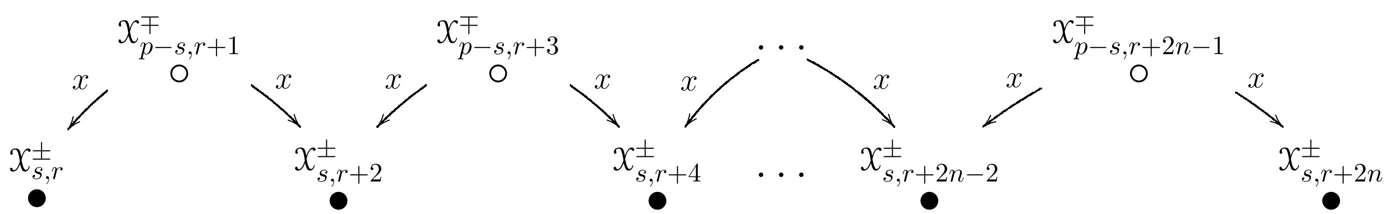
Indecomposable $\mathcal{L}\mathcal{U}_{\mathfrak{q}}s\ell(2)$-modules and Feigin–Fuchs modules.

$\boldsymbol{\mathcal{W}^{\pm}_{s,r}(n)}$**:** The module $\mathcal{W}^{\pm}_{s,r}(n)$ has the following subquotient structure



where n is the number of the bottom modules (filled dots $\bullet$).

$\boldsymbol{\mathcal{M}^{\pm}_{s,r}(n)}$**:** The module $\mathcal{M}^{\pm}_{s,r}(n)$ has the following subquotient structure



where n is the number of the top modules (open dots $\circ$). The $\mathcal{M}^{\pm}_{s,r}(n)$ modules are contragredient to the $\mathcal{W}^{\pm}_{s,r}(n)$ modules.

Indecomposable $\mathcal{L}\mathcal{U}_{\mathfrak{q}}s\ell(2)$-modules and Feigin–Fuchs modules.

Irreducible modules are identified in the following way

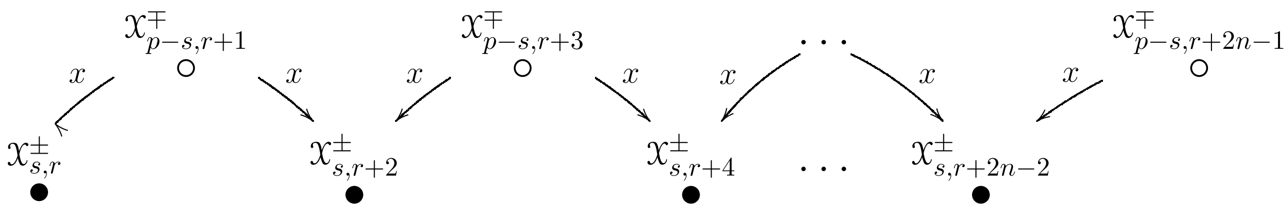
$$\mathcal{X}^{+}_{s,2r-1} \to (2r-1, s), \qquad \mathcal{X}^{-}_{s,2r} \to (2r, s),$$

where (r, s) are the irreducible Virasoro modules with the heighest weights

$$\Delta_{r,s} = ((pr-s)^2 - (p-1)^2)/4p.$$

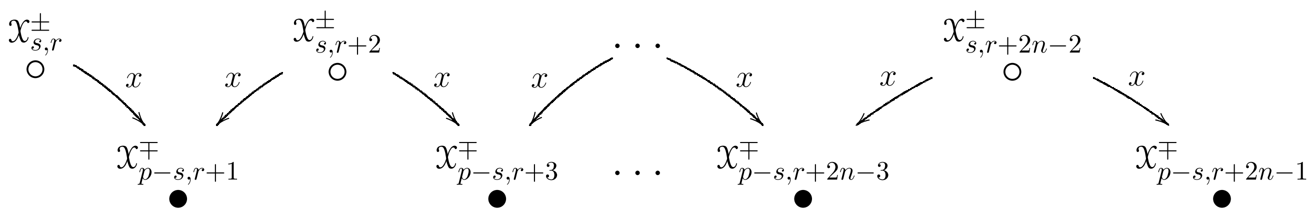
Indecomposable $\mathcal{L}\mathcal{U}_{\mathfrak{q}}s\ell(2)$-modules and Feigin–Fuchs modules.

$\mathcal{N}^{\pm}_{s,r}(n)$**:** The module $\mathcal{N}^{\pm}_{s,r}(n)$ has the following subquotient structure



where n is the number of the top modules (open dots $\circ$) and at the same time the number of the bottom modules (filled dots $\bullet$).

$\overline{\mathcal{N}}^{\pm}_{s,r}(n)$**:** The module $\overline{\mathcal{N}}^{\pm}_{s,r}(n)$ has the following subquotient structure

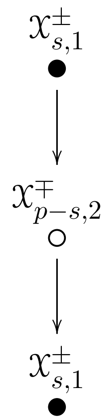


where n is the number of the bottom modules (filled dots $\bullet$) and at the same time the number of the top modules (open dots $\circ$).

The introduced four infinite series of indecomposable modules $\mathcal{W}^{\pm}_{s,r}(n)$, $\mathcal{M}^{\pm}_{s,r}(n)$, $\mathcal{N}^{\pm}_{s,r}(n)$, and $\overline{\mathcal{N}}^{\pm}_{s,r}(n)$ can be used in construction of the Felder type resolutions and projective resolutions.

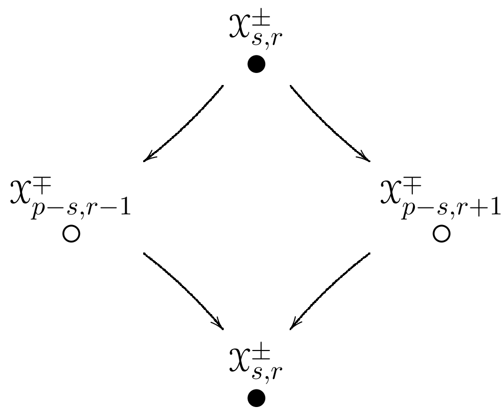
Projective $\mathcal{L}\mathcal{U}_{\mathfrak{q}}s\ell(2)$-modules.

The projective cover $\mathcal{P}^{\pm}_{s,1}$ for the irreducible module $\mathcal{X}^{\pm}_{s,1}$ has the subquotient structure:



Projective $\mathcal{L}\mathcal{U}_{\mathfrak{q}}s\ell(2)$-modules.

The projective cover $\mathcal{P}^{\pm}_{s,r}$ for the irreducible $\mathcal{X}^{\pm}_{s,r}$ has the subquotient structure:



Conclusions.

Relations to Virasoro fusion algebra:

- We identify $\mathcal{L}\mathcal{U}_{\mathfrak{q}} s\ell(2)$ irreducible and projective modules with irreducible and logarithmic modules of the Virasoro algebra $\mathcal{V}_p$.
- Under this identification, tensor products of $\mathcal{L}\mathcal{U}_{\mathfrak{q}} s\ell(2)$-modules coincide with the fusion of the corresponding modules of Gaberdiel and Kausch, and from Pearce and Rasmussen works; and also from recent works of Read and Saleur.
- There exists a tensor functor from "our" category to the category of $\mathcal{V}_p$-modules with dimension of L_0 Jordan cells not greater than 2.

Conclusions.

Relations to Virasoro fusion algebra:

- There exists a tensor functor from "our" category to the category of $\mathcal{V}_p$-modules with dimension of L_0 Jordan cells not greater than 2.
- The functor establishes a 1-to-1 correspondence between simple objects of two categories but **is not** an equivalence because the Virasoro category contains more indecomp objects. In particular, Virasoro Verma modules have no counterpart on the QG side; $\mathcal{V}_p$ also admits a class of modules with two dimensional L_0 Jordan cells enumerated by a projective parameter. All these modules have the same subquotient structure nevertheless are parawise different and only modules with a special value of the parameter has a counterpart on the QG side.